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ON THE BETTI NUMBERS OF ORIENTED GRASSMANNIANS AND INDEPENDENT SEMI-INVARIANTS OF BINARY FORMS

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Dedicated to Professor James D. Stasheff on his 80th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We present a complete functional formula expressing the ith $\mathbb{Z}_2$-Betti number of the oriented Grassmann manifold of oriented 3-dimensional vector subspaces in Euclidean n-space for i from the range determined by the characteristic rank of the canonical oriented 3-dimensional vector bundle over this manifold. The same formula explicitly exhibits the number of linearly independent semi-invariants of degree 3 of a binary form of degree $n-3$. Using the approach and data presented in this note, analogous results can be obtained for the oriented Grassmann manifold of oriented 4-dimensional vector subspaces in Euclidean n-space and semi-invariants of degree 4 of a binary form of degree $n-4$.

1. Introduction and some preliminaries

The $\mathbb{Z}_2$-Betti numbers $b_j(\widetilde{G}_{n,k})$ of the oriented Grassmann manifolds $\widetilde{G}_{n,k}$ of oriented k-dimensional vector subspaces in $\mathbb{R}^n$ are known for $k = 1$ (spheres) and $k = 2$ (complex quadrics); but they are in general unknown for $k \geq 3$. We note that, in contrast to this, the rational Betti numbers of $\widetilde{G}_{n,k}$ are known (thanks to the rational Poincaré polynomial [4: p. 494–495]). In what follows, the cohomology will be taken with coefficients in $\mathbb{Z}_2$; in particular, by the *Betti numbers* we shall always mean the $\mathbb{Z}_2$*-Betti numbers*.

The manifold $\widetilde{G}_{n,k}$ is a double covering space for the Grassmann manifold $G_{n,k}$ of all k-dimensional vector subspaces in $\mathbb{R}^n$; the covering $p\colon \widetilde{G}_{n,k} \to G_{n,k}$ is universal if $(n,k) \neq (2,1)$. In view of the obvious diffeomorphisms $G_{n,k} \cong G_{n,n-k}$, $\widetilde{G}_{n,k} \cong \widetilde{G}_{n,n-k}$, we may assume that $k \leq n-k$.

Recall ([2]) that the cohomology algebra $H^*(G_{n,k})$ of the Grassmann manifold $G_{n,k}$ is generated by the Stiefel-Whitney characteristic classes $w_i(\gamma_{n,k}) \in H^i(G_{n,k})$ of the canonical k-dimensional vector bundle $\gamma_{n,k}$ over $G_{n,k}$. An exact description of the algebra $H^*(G_{n,k})$ is known ([2]) but, for our purposes, it suffices to note that there are no polynomial relations among the generators $w_i(\gamma_{n,k})$ in degrees $\leq n-k$. For each j, the Betti number $b_j(G_{n,k})$ is the same as the number of j-dimensional Schubert cells in $G_{n,k}$, that is ([9: §6]), the number $p(n-k,k,j)$ of restricted partitions of j into at most k parts, each less than or equal to $n-k$; note that $p(n-k,k,j) = p(k,n-k,j)$.

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The Betti number $b_j(G_{n,k})$ is the coefficient at x^j in the Poincaré polynomial ([2])

$$P_x(G_{n,k}) = \frac{(1-x^{n-k+1})(1-x^{n-k+2})\cdots(1-x^n)}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)}. \tag{1.1}$$

As a basis for $H^j(G_{n,k})$ ([5: Theorem, p. 232]), we can take the set

$$\Big\{ w_1^{a_1}\cdots w_k^{a_k}; \sum_{i=1}^{k} i a_i = j, \sum_{i=1}^{k} a_i \le n-k \Big\}, \tag{1.2}$$

where w_i is an abbreviation for the Stiefel-Whitney class $w_i(\gamma_{n,k})$. The set (1.2) will be called the *standard basis* for $H^j(G_{n,k})$.

About the cohomology algebra $H^*(\widetilde{G}_{n,k})$ of the oriented Grassmann manifold $\widetilde{G}_{n,k}$ very little is known in general (see [13]). But it is clear that the pullback $p^*(\gamma_{n,k})$ of the canonical k-plane bundle $\gamma_{n,k}$ over $G_{n,k}$ is the canonical oriented k-plane bundle $\widetilde{\gamma}_{n,k}$ over $\widetilde{G}_{n,k}$, and that

$$\mathrm{Im}(p^* : H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})),$$

multiplicatively generated by the Stiefel-Whitney classes

$$w_i(\widetilde{\gamma}_{n,k}) = p^*(w_i(\gamma_{n,k})),$$

is a self-annihilating subspace, of half the dimension (in general unknown up to now), in the cohomology algebra $H^*(\widetilde{G}_{n,k})$. Thus there exists a positive integer κ ($\kappa \le k(n-k) = \dim(\widetilde{G}_{n,k})$) such that there is some element other than a polynomial in the Stiefel-Whitney classes of $\widetilde{\gamma}_{n,k}$ in the cohomomology group $H^{\kappa+1}(\widetilde{G}_{n,k})$, while all the elements in $H^j(\widetilde{G}_{n,k})$ for nonnegative integers $j \le \kappa$ can be expressed as polynomials in the Stiefel-Whitney classes of $\widetilde{\gamma}_{n,k}$. The number κ is called the *characteristic rank* of $\widetilde{\gamma}_{n,k}$, denoted $\mathrm{charrank}(\widetilde{\gamma}_{n,k})$; for this terminology and further information on the characteristic rank of vector bundles and manifolds see for instance [6], [8], [10], [1], [7]. Now there are various ways (among them a simple analysis of the Gysin exact sequence [9: Corollary 12.3] associated with the double covering p) to verify that, for $l+1 \le \mathrm{charrank}(\widetilde{\gamma}_{n,k})$, one has

$$b_{l+1}(\widetilde{G}_{n,k}) = b_{l+1}(G_{n,k}) - b_l(G_{n,k}); \tag{1.3}$$

the later difference clearly equals the coefficient at x^{l+1} in $(1-x)P_x(G_{n,k})$ (see 1.1). Thanks to Poincaré duality, if we calculate the Betti numbers $b_{l+1}(\widetilde{G}_{n,k})$ for $l+1 \le \frac{k(n-k)}{2}$, then we also have the remaining Betti numbers. We thus confine our attention to those differences $b_{l+1}(G_{n,k}) - b_l(G_{n,k})$, denoted $N(n-k,k,l+1)$ in the sequel, such that

$$l+1 \le \frac{k(n-k)}{2}. \tag{1.4}$$

Thus, for

$$l+1 \le \min\Big\{ \mathrm{charrank}(\widetilde{\gamma}_{n,k}), \frac{k(n-k)}{2} \Big\},$$

the numbers $N(n-k,k,l+1)$ are the Betti numbers of the oriented Grassmann manifold $\widetilde{G}_{n,k}$. But additionally, by Cayley - Sylvester's theorem, $N(n-k,k,l+1)$ is also equal to the number of linearly independent semi-invariants of degree k and weight $l+1$ of a binary form of degree $n-k$ (see for example [11], [12: Satz 2.21]).

Recently, the characteristic rank of $\widetilde{\gamma}_{n,3}$ has been found for infinitely many values of n. For instance, the following was proved in [7]: for $n \ge 6$, taking c to be the only integer such that

$2^{c-1} < n \le 2^c$, we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \begin{cases} = n-2 & \text{if } n = 2^c, \\ = n-5+i & \text{if } n = 2^c - i,\ i \in \{1, 2, 3\}, \\ \ge n-2 & \text{otherwise.} \end{cases} \tag{1.5}$$

By (1.5), for example, if $l+1 \le 2^c - 2$, then each Betti number $b_{l+1}(\widetilde{G}_{2^c,3})$ equals the number $N(2^c-3, 3, l+1)$.

As the main result, this note presents a complete functional formula for $N(n-3, 3, l+1)$: the result is given in Theorem 2.2, by Table 1. Via our formula, one gains new, global insight into the ith Betti numbers of $\widetilde{G}_{n,3}$ for $i \le \min\{\operatorname{charrank}(\widetilde{\gamma}_{n,3}), \frac{3(n-3)}{2}\}$. The same formula explicitly exhibits the number of linearly independent semi-invariants of degree 3 of a binary form of degree $n-3$. In particular, Bundy and Hart's [3: Theorem 3.1(b)] is immediately obtained from our formula; see Remark 3.

In order to keep the present note within reasonable size, we confine ourselves to noting that, using the approach and data presented in this note, analogous results (extending [3: Theorem 3.1(c)]) can be obtained for the oriented Grassmann manifold of oriented 4-dimensional vector subspaces in Euclidean n-space and linearly independent semi-invariants of degree 4 of a binary form of degree $n-4$. We add that the situation for $N(n-2, 2, l+1)$ is very simple; we shall see, in Lemma 2.1, that $N(n-2, 2, l+1) = 0$ if l is even, and $N(n-2, 2, l+1) = 1$ if l is odd; Lemma 2.1 implies [3: Theorem 3.1(a)].

2. Additional preliminaries and the main result

We first give a recursive formula for the Betti numbers of the Grassmann manifold $G_{n,k}$. As starting data, one has that the ith Betti number of the real $(n-1)$-dimensional projective space $G_{n,1}$ is equal to 1 for $i = 0, 1, \dots, n-1$ and, of course, it vanishes for all other values of i.

Proposition 2.1. *For the Betti numbers of the Grassmann manifold $G_{n,k}$ $(2 \le k \le n-k)$, we have the following recursion:*

$$b_l(G_{n,k}) = \sum_{0 \le i \le \frac{l}{k}} b_{l-ik}(G_{n-1-i,k-1}).$$

Proof. The standard basis for $H^l(G_{n,k})$ is the union of pairwise disjoint subsets

$$P_i = \Big\{ w_1^{a_1} \cdots w_{k-1}^{a_{k-1}} w_k^i;\ \sum_{j=1}^{k-1} j a_j + ik = l,\ \sum_{j=1}^{k-1} a_j + i \le n-k \Big\},$$

where $i = 0, 1, \dots, \lfloor \frac{l}{k} \rfloor$. But the elements of the set P_i are in an obvious bijective correspondence with the elements of the standard basis for $H^{l-ik}(G_{n-1-i,k-1})$. Thus the cardinality of P_i is the Betti number $b_{l-ik}(G_{n-1-i,k-1})$, and the proposition is proved. □

As an immediate application of the recursive formula from Proposition 2.1, we obtain the following.

Lemma 2.1. *Let $n \ge 4$.*

(a) *If $0 \le l \le n-2$, then we have that $N(n-2, 2, l) = b_l(G_{n,2}) - b_{l-1}(G_{n,2})$ is equal to 0 if l is odd, and is equal to 1 if l is even.*

(b) *If $n-2 < l \le 2(n-2)$, then we have that $b_l(G_{n,2}) - b_{l-1}(G_{n,2})$ is equal to -1 if l is odd, and is equal to 0 if l is even.*

TABLE 1. A complete formula for $N(n-3,3,l+1)$

$j\backslash x$	*0*	*1*	*2*	*3*
0	$\delta-1(\geq 1)$ *if* $\varepsilon\geq 0$; $2t$ *otherwise*	$\delta(\geq 0)$ *if* $\varepsilon\geq 1$; $2t$ *otherwise*	$\delta(\geq 0)$ *if* $\varepsilon\geq 1$; $2t$ *otherwise*	$\delta+1(\geq 1)$ *if* $\varepsilon\geq 2$; $2t$ *otherwise*
1	$\delta(\geq 2)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*	$\delta(\geq 2)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*	$\delta+1(\geq 1)$ *if* $\varepsilon\geq 1$; $2t+1$ *otherwise*	$\delta+1(\geq 1)$ *if* $\varepsilon\geq 1$; $2t+1$ *otherwise*
2	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -1$; $2t+1$ *otherwise*	$\delta(\geq 2)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*	$\delta(\geq 0)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*	$\delta+1(\geq 1)$ *if* $\varepsilon\geq 1$; $2t+1$ *otherwise*
3	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -1$; $2t+1$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -1$; $2t+1$ *otherwise*	$\delta(\geq 2)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*	$\delta(\geq 0)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*
4	$\delta-2(\geq 0)$ *if* $\varepsilon\geq -2$; $2t+1$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -1$; $2t+1$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -1$; $2t+1$ *otherwise*	$\delta(\geq 2)$ *if* $\varepsilon\geq 0$; $2t+1$ *otherwise*
5	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -2$; $2t+2$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -2$; $2t+2$ *otherwise*	$\delta(\geq 2)$ *if* $\varepsilon\geq -1$; $2t+2$ *otherwise*	$\delta(\geq 2)$ *if* $\varepsilon\geq -1$; $2t+2$ *otherwise*
6	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -3$; $2t+1$ *otherwise*	$\delta-2(\geq 0)$ *if* $\varepsilon\geq -2$; $2t+1$ *otherwise*	$\delta-2(\geq 0)$ *if* $\varepsilon\geq -2$; $2t+1$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -1$; $2t+1$ *otherwise*
7	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -2$; $2t+2$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -2$; $2t+2$ *otherwise*
8	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -4$; $2t+2$ *otherwise*	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*	$\delta-2(\geq 0)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*	$\delta-1(\geq 1)$ *if* $\varepsilon\geq -2$; $2t+2$ *otherwise*
9	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -4$; $2t+2$ *otherwise*	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -4$; $2t+2$ *otherwise*	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*	$\delta-2(\geq 0)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*
10	$\delta-4(\geq 0)$ *if* $\varepsilon\geq -5$; $2t+2$ *otherwise*	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -4$; $2t+2$ *otherwise*	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -4$; $2t+2$ *otherwise*	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -3$; $2t+2$ *otherwise*
11	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -5$; $2t+3$ *otherwise*	$\delta-3(\geq 1)$ *if* $\varepsilon\geq -5$; $2t+3$ *otherwise*	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -4$; $2t+3$ *otherwise*	$\delta-2(\geq 2)$ *if* $\varepsilon\geq -4$; $2t+3$ *otherwise*

Proof. Part (a). By Proposition 2.1, we have that

$$N(n-2,2,l)=\sum_{0\leq i\leq \frac{l}{2}} b_{l-2i}(G_{n-1-i,1})-\sum_{0\leq i\leq \frac{l-1}{2}} b_{l-1-2i}(G_{n-1-i,1})=\sum_{0\leq i\leq \frac{l}{2}} 1_i-\sum_{0\leq i\leq \frac{l-1}{2}} 1_i,$$

where $1_i=1$ for all i. This proves Part (a).
Part (b). By Poincaré duality, we have

$$b_l(G_{n,2})-b_{l-1}(G_{n,2})=b_{2(n-2)-l}(G_{n,2})-b_{2(n-2)-l+1}(G_{n,2})=-(b_{2(n-2)-l+1}(G_{n,2})-b_{2(n-2)-l}(G_{n,2})).$$

By applying Part (a), we obtain the result. The lemma is proved. □

Remark 1. We note that Lemma 2.1(a) obviously implies [3: Theorem 3.1(a)].

The following is the main result of the present note: a complete functional formula expressing the number $N(n-3,3,l+1)$.

THEOREM 2.2. *Let* $l=12t+j$ *with* $j=0,1,2,\ldots,11$ *and* $n-3=4s+x$ *with* $x=0,1,2,3$, *where* $s\geq 1$ *if* $x=0,1,2$ *and* $s\geq 0$ *if* $x=3$. *For typographical reasons, we abbreviate* $\delta:=2s-4t$ *and* $\varepsilon:=6t-2s$. *By the definition of* $N(n-3,3,l+1)$, *we suppose (see (1.4)) that* $\delta\geq\frac{2j-3x+2}{6}$. *Then* Table 1 *presents a complete functional formula for the number* $N(n-3,3,l+1)$ $(n\geq 6)$.

Remark 2. For reasons of space we write the data in Table 1 in a somewhat condensed way. For instance, $\begin{matrix}\delta-1_{(\geq 1)\text{ if }\varepsilon\geq 0;}\\ 2t_{\text{ otherwise}}\end{matrix}$ appearing in the upper left-hand corner is to be read: for $j=0$ and $x=0$, we have $N(n-3,3,l+1)=\delta-1$ (which is ≥ 1) if $\varepsilon\geq 0$, and we have $N(n-3,3,l+1)=2t$ if $\varepsilon<0$.

P r o o f. By Proposition 2.1, we have that

$$N(n-3,3,l+1)=b_{12t+j+1}(G_{4s+x+3,3})-b_{12t+j}(G_{4s+x+3,3})$$

is equal to

$$\sum_{i=0}^{\lfloor 4t+\frac{j+1}{3}\rfloor} b_{12t+j+1-3i}(G_{4s+x+2-i,2})-\sum_{i=0}^{\lfloor 4t+\frac{j}{3}\rfloor} b_{12t+j-3i}(G_{4s+x+2-i,2}). \tag{2.1}$$

Now it suffices to apply Lemma 2.1. To illustrate this, we present two cases in detail.

Case $j=0,\ x=0$. By the definition of the number $N(n-3,3,l+1)$, our assumption now is that $\delta\geq\frac{1}{3}$, that is (note that δ is always even), $\delta\geq 2$. By (2.1), we see that $N(n-3,3,l+1)$ is equal to

$$\sum_{i=0}^{4t} b_{12t+1-3i}(G_{4s+2-i,2})-b_{12t-3i}(G_{4s+2-i,2}).$$

To apply Lemma 2.1, we need to know for which i one has $12t+1-3i\leq 4s-i$. Of course, this is the case precisely for $i\geq 6t-2s+1$, that is, $i\geq\varepsilon+1$. Thus Lemma 2.1 implies that

$$N(n-3,3,l+1)=\sum_{0\leq\text{ even }i\leq\varepsilon}(-1)_i+\sum_{\varepsilon+1\leq\text{ non-negative odd }i\leq 4t}(+1)_i;$$

the right-hand side is equal to $(-1)(3t-s+1)+(+1)(s-t)=2s-4t-1=\delta-1$ (which is ≥ 1) if $\varepsilon\geq 0$, and is equal to $2t$ if $\varepsilon<0$ (of course, we understand that $(-1)_i=-1$ and $(+1)_i=1$ for all i). This proves the claim for $j=0,\ x=0$.

Case $j=2,\ x=1$. By the definition of $N(n-3,3,l+1)$, our assumption now is that $\delta\geq 2$. By (2.1), we see that $N(n-3,3,l+1)$ is equal to

$$b_0(G_{4(s-t)+2,2})+\sum_{i=0}^{4t} b_{12t+3-3i}(G_{4s+3-i,2})-b_{12t+2-3i}(G_{4s+3-i,2}).$$

As is well known, $b_0(G_{4(s-t)+2,2})=1$. We need to know for which i one has $12t+3-3i\leq 4s+1-i$. Clearly, this is the case for $i\geq\varepsilon+1$. Thus Lemma 2.1 implies that

$$N(n-3,3,l+1)=1+\sum_{0\leq\text{ even }i\leq\varepsilon}(-1)_i+\sum_{\varepsilon+1\leq\text{ non-negative odd }i\leq 4t}(+1)_i;$$

the right-hand side is equal to $1+(-1)(3t-s+1)+(+1)(s-t)=2s-4t=\delta$ (which is ≥ 2) if $\varepsilon\geq 0$, and is equal to $1+2t$ if $\varepsilon<0$. This proves the claim for $j=2,\ x=1$.

All the remaining cases are done similarly. The proof of Theorem 2.2 is finished. □

Remark 3. From Table 1, one readily sees precisely when $N(n-3,3,l+1)=b_{l+1}(G_{n,3})-b_l(G_{n,3})=p(n-3,3,l+1)-p(n-3,3,l)$ vanishes. To verify more easily that Bundy and Hart's [3: Theorem 3.1(b)] (answering the question of when $p(n-3,3,l+1)=p(n-3,3,l)$ and, from the topological viewpoint, also determining all those numbers $i+1\leq\min\{\operatorname{charrank}(\widetilde{\gamma}_{n,3}),\frac{3(n-3)}{2}\}$ such that $H^{i+1}(\widetilde{G}_{n,3})=0$) is immediately obtained from our Theorem 2.2, we put $n-3=q$. One readily checks that we have $N(q,3,l+1)=0$ precisely in the following cases:

(1) $j = 0$ and $t = 0$ (these are the obvious cases of $b_1(G_{n,3}) - b_0(G_{n,3}) = 1 - 1 = 0$);
(2) $j = 0$, $x = 1$, $\delta = 0$, $\varepsilon \geq 1$; this means that $q = 8t + 1$ ($t \geq 1$), $l = \frac{3q-3}{2}$;
(3) $j = 0$, $x = 2$, $\delta = 0$, $\varepsilon \geq 1$; this means that $q = 8t + 2$ ($t \geq 1$), $l = \frac{3q-6}{2}$;
(4) $j = 2$, $x = 2$, $\delta = 0$, $\varepsilon \geq 0$; this means that $q = 8t + 2$ ($t \geq 1$), $l = \frac{3q-2}{2}$;
(5) $j = 3$, $x = 3$, $\delta = 0$, $\varepsilon \geq 0$; this means that $q = 8t + 3$ ($t \geq 0$), $l = \frac{3q-3}{2}$;
(6) $j = 4$, $x = 0$, $\delta - 2 = 0$, $\varepsilon \geq -2$; this means that $q = 8t + 4$ ($t \geq 0$), $l = \frac{3q-4}{2}$;
(7) $j = 6$, $x = 1$, $\delta - 2 = 0$, $\varepsilon \geq -2$; this means that $q = 8t + 5$ ($t \geq 0$), $l = \frac{3q-3}{2}$;
(8) $j = 6$, $x = 2$, $\delta - 2 = 0$, $\varepsilon \geq -2$; this means that $q = 8t + 6$ ($t \geq 0$), $l = \frac{3q-6}{2}$;
(9) $j = 8$, $x = 2$, $\delta - 2 = 0$, $\varepsilon \geq -3$; this means that $q = 8t + 6$ ($t \geq 0$), $l = \frac{3q-2}{2}$;
(10) $j = 9$, $x = 3$, $\delta - 2 = 0$, $\varepsilon \geq -3$; this means that $q = 8t + 7$ ($t \geq 0$), $l = \frac{3q-3}{2}$;
(11) $j = 10$, $x = 0$, $\delta - 4 = 0$, $\varepsilon \geq -5$; this means that $q = 8t + 8$ ($t \geq 0$), $l = \frac{3q-4}{2}$.

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